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Two-dimensional travelling waves over a moving bottom

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ABSTRACT

Two-dimensional travelling waves on an ideal fluid with gravity and surface tension over a periodically moving bottom with a small amplitude are studied. The bottom and the wave travel with a same speed. The exact Euler equations are formulated as a spatial dynamic system by using the stream function. A manifold reduction technique is applied to reduce the system into one of ordinary differential equations with finite dimensions. A homoclinic solution to the normal form of this reduced system persists when higher-order terms are added, which gives a generalized solitary wave—the homoclinic solution connecting a periodic solution.

1. Introduction

There are a lot of numeric results involving flows past moving surface-piercing or submerged objects. The flows reduce to ones past disturbances when viewed in a frame of reference moving with the objects. Lamb [25] calculated a flow past a submerged semi-elliptical obstacle by an approximate linear theory. Forbes [13] and Forbes and Schwartz [14] solved the corresponding exact problem numerically. They considered both semicircular and semi-elliptical obstacles. Their results confirmed and extended Lamb's solutions. Dias and Vanden-Broeck computed flows past a submerged triangular obstacle [9] and two submerged obstacles of arbitrary shape [11]. They [10,12] also investigated the motion forced by obstacles in the flow of two contiguous homogeneous fluids of different constant

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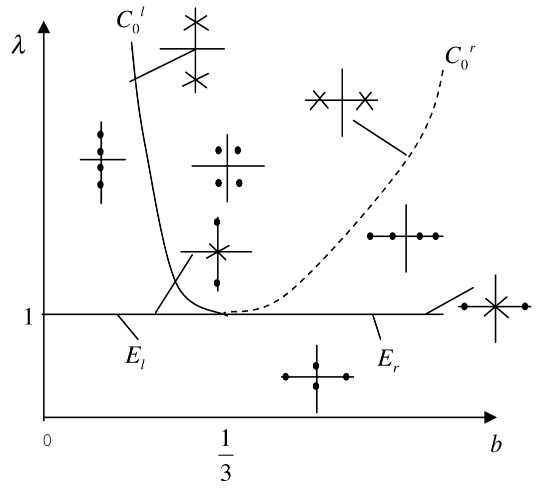

Fig. 1. The curves C_0^l, E_l, E_r consist of points in (b,λ)-parameter space at which the number of purely imaginary eigenvalues of linearized problem changes; at C_0^r four nonzero, real eigenvalues become complex without passing through zero. Dots and crosses denote respectively simple and double eigenvalues.

densities and different thickness. Asavanant and Vanden-Broeck [2] studied flows past a parabolic obstacle lying on the free surface and Binder and Vanden-Broeck [4] considered flows with the free surface past surfboards and sluice gates. There are also some interesting results about the existence of periodic solutions over a moving bottom. For example, see [15,18,19] and [24]. However, the rigorous existence theory of solitary wave solutions with disturbances is much less complete. Sun and Shen [28] investigated a two-dimensional travelling wave over a bump with gravity and a prescribed pressure on the free surface. The bump and the pressure are small and symmetric with compact support. They theoretically proved the existence of a solitary wave as the size of the bump and pressure tends to zero.

The existence of two-dimensional waves without disturbances has been proved rigorously (e.g. see Dias and Iooss [8]). It depends on two dimensionless parameters b and $\lambda = F^{-2}$ where b is the Bond number and F is the Froude number. The distribution of eigenvalues of the liner operator corresponding to this hydrodynamic problem is given in Fig. 1. The curves C_0^l, C_0^r, E_l, E_r are associated with bifurcation phenomena.

Amick and Kirchgässner [1] studied the curve $E_r \setminus \{(\frac{1}{3}, 1)\}$. They found a solitary wave of depression just above this curve and proved it to be unique. Buffoni, Groves and Toland [6] considered the parameter region to the left of the curve C_0^r and near the point $(\frac{1}{3}, 1)$. They proved the existence of infinitely many distinct solitary wave solutions. For the region that lies on the right side of $C_0^l \setminus \{(\frac{1}{3}, 1)\}$, Buffoni and Groves [5], Dias and Iooss [7], Iooss and Kirchgässner [20], and Iooss and Pérouème [22] showed the existence of solitary waves. Beale [3], Iooss and Kirchgässner [21], and Sun [27] gave the existence of generalized solitary waves which have an oscillatory tail of exponentially small amplitude at infinity just below the curve $E_l \setminus \{(\frac{1}{3}, 1)\}$.

Motivated by the numeric results about flows past disturbances and one by Sun and Shen [28]. This paper considers an inviscid, irrotational and incompressible two-dimensional wave subject to gravity and surface tension. It is bounded above by a free surface and below by a bottom periodically moving with a small amplitude. The waves are travelling with the same speed of the bottom. The distribution of eigenvalues of the linearized problem is same as one in Fig. 1 because the bottom has a very small amplitude. We especially focus on E_r while other curves C_0^l, C_0^r, E_l can be similarly dealt with, and prove the existence of a generalized solitary wave which approaches a periodic wave solution in the horizontal direction x at infinity when (b,λ) is above E_r, which is different from the result of the existence of a solitary wave solution approaching zero at infinity by Amick and Kirchgässner [1] since the bottom here is periodically moving. The idea in this paper can also been used for waves past other disturbances like a small bump with compact support.

This paper is organized as follows. In Section 2, we combine the methods from Dias and Iooss [8] and Groves and Wahlen [17] to change the governing equations of the wave problem into a spatial dynamic system by using x and the stream function as independent variables. The properties of its linear operator and a manifold reduction theorem given by Mielke [26] are stated in Section 3. The spatial dynamic system is reduced to a system of ordinary differential equations with finite dimensions. The adjoint operator of this linear operator is also given. Section 4 studies the case: (b, λ) near the curve E_r. There is a double eigenvalue zero while other eigenvalues have nonzero real parts. The reduced system consists of two ordinary differential equations. Its normal form is also given. Section 5 proves the existence of a generalized solitary wave. The idea is similar to one by Groves and Mielke [16]. Firstly, the existence of a periodic solution is given following the method in the book [23] by Kielhöfer, which will be used for the behavior of a generalized solitary wave at infinity. The normal form of the reduced system has a homoclinic orbit which persists when the higher-order terms are included by using a fixed point theorem and a perturbation method. This gives the existence of a generalized solitary wave approaching a periodic solution at infinity.

2. Formulation as a spatial dynamical system

A two-dimensional fluid here is assumed to be inviscid, irrotational and incompressible subject to gravity and surface tension. It is bounded above by a free surface $y = d + \eta(t, x)$ and below by a moving bottom $y = s^*(t, x)$ where $\eta > -d$ and d represents a reference depth of the fluid at infinity. Suppose that the bottom is periodically travelling with a constant speed c and a small amplitude $\epsilon > 0$, i.e. $s^*(t, x) = \epsilon s(x - ct)$ where s is a smooth and periodic function. We are interested in the travelling wave with the same speed c so that $\eta(t, x) = \eta(x - ct)$. Let $D_{\epsilon,\eta} = \{(x, y)\colon\ x \in \mathbf{R},\ \epsilon s < y < d + \eta\}$, (u, v) and ψ denote the fluid domain, the velocity and the stream function respectively. Then $\psi_x = -v$ and $\psi_y = u - c$. In this paper, we suppose

$$\psi_y = u - c \leqslant -\delta < 0 \quad \text{in the closure } \bar{D}_{\epsilon,\eta} \tag{2.1}$$

where $\delta > 0$ is a constant. Assume that $\psi = 0$ on the free surface $y = d+\eta$ and $\psi = m_0$ on the bottom $y = \epsilon s$ where $m_0 > 0$. The hydrodynamic problem (e.g. see [17]) is to solve the elliptic equation

$$\psi_{xx} + \psi_{yy} = 0 \quad \text{for } \epsilon s < y < d + \eta \tag{2.2}$$

subject to the boundary conditions

$$\psi\big(x, d + \eta(x)\big) = 0, \tag{2.3}$$

$$\psi\big(x, \epsilon s(x)\big) = m_0, \tag{2.4}$$

$$\frac{1}{2}\big(\psi_x^2 + \psi_y^2\big) - \kappa\left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}}\right)_x + g\eta = Q \quad \text{on } y = d + \eta \tag{2.5}$$

where κ is the coefficient of surface tension, g is the constant gravitational acceleration, and Q is the Bernoulli constant.

Now we rewrite Eqs. (2.2)–(2.5) in terms of the dimensionless variables

$$(\tilde{x}, \tilde{y}) = \frac{1}{d}(x, y), \qquad \tilde{\eta}(\tilde{x}) = \frac{1}{d}\eta(x), \qquad \tilde{s}(\tilde{x}) = \frac{1}{d}s(x), \qquad \tilde{\psi}(\tilde{x}, \tilde{y}) = -\frac{1}{m_0}\psi$$

so we have

$$\tilde{\psi}_{\tilde{x}\tilde{x}} + \tilde{\psi}_{\tilde{y}\tilde{y}} = 0 \quad \text{for } \epsilon\tilde{s} < \tilde{y} < 1 + \tilde{\eta}, \tag{2.6}$$

$$\tilde{\psi}\big(\tilde{x}, 1 + \tilde{\eta}(\tilde{x})\big) = 0, \tag{2.7}$$

$$\tilde{\psi}\big(\tilde{x}, \epsilon\tilde{s}(\tilde{x})\big) = -1, \tag{2.8}$$

$$\frac{1}{2}\big(\tilde{\psi}_{\tilde{x}}^2 + \tilde{\psi}_{\tilde{y}}^2\big) - b\left(\frac{\tilde{\eta}_{\tilde{x}}}{\sqrt{1 + \tilde{\eta}_{\tilde{x}}^2}}\right)_{\tilde{x}} + \lambda\tilde{\eta} = \rho_0 \quad \text{on } \tilde{y} = 1 + \tilde{\eta} \tag{2.9}$$

where

$$b = \frac{\kappa d}{m_0^2}, \qquad \lambda = \frac{gd^3}{m_0^2}, \qquad \rho_0 = \frac{Q\,d^2}{m_0^2}. \tag{2.10}$$

The next step is to map the unknown fluid domain $D_{\epsilon,\eta}$ into a fixed strip $\mathbf{R} \times (0, 1)$ using x and the stream function ψ as independent variables (e.g. see [17]). Let

$$\tau = -\tilde{\psi}, \qquad h = \tilde{y}$$

which change (2.6)–(2.9) into

$$\frac{\partial}{\partial\tilde{x}}\left(\frac{h_{\tilde{x}}}{h_\tau}\right) - \left(\frac{1 + h_{\tilde{x}}^2}{2h_\tau^2}\right)_\tau = 0 \quad \text{for } 0 < \tau < 1, \tag{2.11}$$

$$\frac{1 + h_{\tilde{x}}^2}{2h_\tau^2} + \lambda\tilde{\eta} - b\left(\frac{\tilde{\eta}_{\tilde{x}}}{\sqrt{1 + \tilde{\eta}_{\tilde{x}}^2}}\right)_{\tilde{x}} = \rho_0 \quad \text{on } \tau = 0, \tag{2.12}$$

$$h(\tilde{x}, 1) = \epsilon\tilde{s}, \tag{2.13}$$

$$h(\tilde{x}, 0) = 1 + \tilde{\eta}. \tag{2.14}$$

Note that (2.1) implies $h_\tau > 0$. Now, using the methods from [8] and [17], we transform Eqs. (2.11)–(2.14) in a spatial dynamical system. Let

$$\hat{h} = h - \epsilon\tau^2\tilde{s}, \qquad u = \frac{h_{\tilde{x}}}{h_\tau}, \qquad \xi = \frac{\tilde{\eta}_{\tilde{x}}}{\sqrt{1 + \tilde{\eta}_{\tilde{x}}^2}} \tag{2.15}$$

which yield from (2.11)–(2.14)

$$\begin{aligned}
\hat{h}_{\tilde{x}} &= u(\hat{h}_\tau + 2\epsilon\tau\tilde{s}) - \epsilon\tau^2\tilde{s}', \\
u_{\tilde{x}} &= \left(\frac{1 + u^2(\hat{h}_\tau + 2\epsilon\tau\tilde{s})^2}{2(\hat{h}_\tau + 2\epsilon\tau\tilde{s})^2}\right)_\tau, \\
\tilde{\eta}_{\tilde{x}} &= \frac{\xi}{\sqrt{1 - \xi^2}}, \\
\xi_{\tilde{x}} &= \left[\frac{1 + u^2\hat{h}_\tau^2}{2b\hat{h}_\tau^2} + \frac{\lambda}{b}\tilde{\eta}\right]\bigg|_{\tau=0} - \frac{\rho_0}{b}
\end{aligned} \tag{2.16}$$

with boundary conditions

$$\begin{aligned}
&\hat{h}(\tilde{x}, 0) = \tilde{\eta} + 1, \\
&\hat{h}(\tilde{x}, 1) = 0, \\
&u\hat{h}_\tau|_{\tau=0} = \frac{\xi}{\sqrt{1-\xi^2}}, \\
&\big(u(\hat{h}_\tau + 2\epsilon\tilde{s}) - \epsilon\tilde{s}'\big)\big|_{\tau=1} = 0.
\end{aligned} \tag{2.17}$$

Here the last two equations of (2.17) can be deduced from the first two equations of (2.17) and the first and third equations of (2.16).

It is easy to check that $(\hat{h}, u, \tilde{\eta}, \xi)^T = (1-\tau, 0, 0, 0)^T$ is an equilibrium of the system (2.16) with boundary conditions (2.17) when $\epsilon = 0$ and $\rho_0 = \frac{1}{2}$. Note that the last two boundary conditions in (2.17) are nonlinear. In order to change them into linear ones and move the equilibrium to the origin, let

$$\tilde{h} = \hat{h} - (1-\tau), \qquad v = u(\hat{h}_\tau + 2\epsilon\tau\tilde{s}) - \epsilon\tau^2\tilde{s}', \qquad \varrho = \frac{\xi}{\sqrt{1-\xi^2}},$$

which change (2.16) and (2.17) into (dropping the tilde and the hat)

$$\begin{aligned}
&h_x = v, \\
&v_x = (h_\tau - 1 + 2\epsilon\tau s)\left(\frac{1 + (v + \epsilon\tau^2 s')^2}{2(h_\tau - 1 + 2\epsilon\tau s)^2}\right)_\tau - \epsilon\tau^2 s'' + \frac{(v + \epsilon\tau^2 s')(v_\tau + 2\epsilon\tau s')}{h_\tau - 1 + 2\epsilon\tau s}, \\
&\eta_x = \varrho, \\
&\varrho_x = \big(1+\varrho^2\big)^{3/2}\left(\frac{1+\varrho^2}{2b(h_\tau - 1)^2}\bigg|_{\tau=0} + \frac{\lambda}{b}\eta - \frac{1}{2b}\right)
\end{aligned} \tag{2.18}$$

with boundary conditions

$$h(x, 0) = \eta, \qquad h(x, 1) = 0, \qquad v(x, 0) = \varrho, \qquad v(x, 1) = 0. \tag{2.19}$$

Symbolically, write the system (2.18) as

$$\dot{w} = \mathcal{F}(x, w; \epsilon, \lambda, b) \tag{2.20}$$

where $w = (h, v, \eta, \varrho)^T$, $\mathcal{F}$ has the following property

$$\mathcal{F}(x, w; 0, \lambda, b) = \mathcal{F}_0(w; \lambda, b) \tag{2.21}$$

and $\mathcal{F}_0(w; \lambda, b)$ does not explicitly include x.

Make the following assumption:

(A) s in (2.20) is $(k+1)$-times continuously differentiable on $\mathbf{R}$, even and periodic with a period P where k is any positive integer.

Then the system (2.20) is reversible with a reverser S defined by

$$S(h, v, \eta, \varrho) = (h, -v, \eta, -\varrho), \tag{2.22}$$

that is, $S(h, v, \eta, \varrho)(-x)$ is also a solution whenever $(h, v, \eta, \varrho)(x)$ is. A solution (h, v, η, ϱ) is reversible if $S(h, v, \eta, \varrho)(-x) = (h, v, \eta, \varrho)(x)$. Clearly, $\mathcal{F}$ satisfies

$$S\mathcal{F}(x, w; \epsilon, \lambda, b) = -\mathcal{F}(-x, Sw; \epsilon, \lambda, b). \tag{2.23}$$

Define the Hilbert spaces

$$\begin{gathered} Y = \big\{(h, v, \eta, \varrho) \in H^2(0,1) \times H^1(0,1) \times \mathbf{R} \times \mathbf{R},\ h(0) = \eta,\ h(1) = 0\big\}, \\ X = \big\{(h, v, \eta, \varrho) \in H^1(0,1) \times L^2(0,1) \times \mathbf{R} \times \mathbf{R},\ h(0) = \eta,\ h(1) = 0\big\}, \\ Z = \big\{(h, v, \eta, \varrho) \in H^2(0,1) \times H^1(0,1) \times \mathbf{R} \times \mathbf{R},\ h(0) = \eta,\ h(1) = 0,\ v(0) = \varrho,\ v(1) = 0\big\}. \end{gathered} \tag{2.24}$$

Clearly, $\mathcal{F}(x, \cdot; \epsilon, \lambda, b)$ maps Y to X. Rewrite the system (2.20) as

$$\dot{w} = Aw + \mathcal{F}_1(x, w; \epsilon, \lambda, b) \tag{2.25}$$

where the linear operator A with a domain $\mathcal{D}(A) = Z$ is the derivative of $\mathcal{F}_0$ in (2.21) with respect to w at $w = 0$, which is given by

$$Aw = \begin{pmatrix} v \\ -h_{\tau\tau} \\ \varrho \\ \frac{1}{b} h_\tau|_{\tau=0} + \frac{\lambda}{b}\eta \end{pmatrix}, \tag{2.26}$$

and $\mathcal{F}_1(x, w; \epsilon, \lambda, b) = \mathcal{F}(x, w; \epsilon, \lambda, b) - Aw$.

3. Properties of linear operator and manifold reduction theorem

This section yields the properties of the linear operator A. Then applying a manifold reduction theorem [26] reduces the system (2.25) into one of differential equations with finite dimensions.

Lemma 3.1.

(1) *There exist constants C and $\sigma_0 > 0$ such that each solution $w \in \mathcal{D}(A) = Z$ of the resolvent equation*

$$(A - i\sigma I)w = w^*, \tag{3.1}$$

where w^* belongs to X and σ is a real number with $|\sigma| \geqslant \sigma_0$, satisfies

$$\|w\|_Y \leqslant C\|w^*\|_X, \tag{3.2}$$

$$\|w\|_X \leqslant \frac{C}{|\sigma|}\|w^*\|_X. \tag{3.3}$$

(2) *The complex number $\sigma \neq 0$ is an eigenvalue of A if and only if*

$$\big(\lambda - \sigma^2 b\big) - \sigma \coth\sigma = 0. \tag{3.4}$$

For $\lambda = 1$ and $b \neq \frac{1}{3}$, 0 is a double eigenvalue while it is an eigenvalue with multiplicity 4 for $\lambda = 1$ and $b = \frac{1}{3}$. The spectrum $\tilde{\sigma}(A)$ of A consists of isolated eigenvalues of finite algebraic multiplicity and $\tilde{\sigma}(A) \cap i\mathbf{R}$ is a finite set.

The proofs are similar to ones in [21] even though the expressions of the linear operators are different. The distribution of eigenvalues of A is given in Fig. 1 (also see [21]).

We introduce a manifold reduction theorem given by Mielke [26].

Theorem 3.1. *Consider the differential equation*

$$\dot{u} = Ku + N(x, u; \tilde{\lambda}) \tag{3.5}$$

in which u belongs to a Hilbert space $\mathcal{X}$, $\tilde{\lambda} \in \mathbf{R}^n$ is a parameter and $K : \mathcal{D}(K) \subset \mathcal{X} \to \mathcal{X}$ is a densely defined closed linear operator. Regard $\mathcal{D}(K)$ as a Hilbert space equipped with the graph norm, and suppose that 0 *is an equilibrium point of* (3.5) *when $\tilde{\lambda} = 0$ and that the following hypotheses hold.*

(H1) *$\mathcal{X}$ admits a direct-sum decomposition $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$, where $\mathcal{X}_1, \mathcal{X}_2$ are closed K-invariant subspaces, so that* (3.5) *can be written as the linearly decoupled system*

$$\dot{u}_1 = K_1 u_1 + N_1(x, u_1 + u_2; \tilde{\lambda}), \tag{3.6}$$

$$\dot{u}_2 = K_2 u_2 + N_2(x, u_1 + u_2; \tilde{\lambda}) \tag{3.7}$$

in which $K_j = K|_{\mathcal{D}(K)\cap\mathcal{X}_j} : \mathcal{D}(K) \cap \mathcal{X}_j \subset \mathcal{X}_j \to \mathcal{X}_j$, $j = 1, 2$.

(H2) *$\mathcal{X}_1$ is finite dimensional and the spectrum of K_1 lies on the imaginary axis.*

(H3) *The imaginary axis lies in the resolvent set of K_2 and*

$$\left\|(K_2 - iaI)^{-1}\right\| \leqslant \frac{C}{1 + |a|}, \quad a \in \mathbf{R},$$

for some constant C that is independent of a.

(H4) *There exist a natural number k and neighborhoods $\Lambda \subset \mathbf{R}^n$ of* 0 *and $U \subset \mathcal{D}(K)$ of* 0 *such that N is $(k+1)$-times continuously differentiable on $\mathbf{R} \times U \times \Lambda$, it and its derivatives are bounded and uniformly continuous on $\mathbf{R} \times U \times \Lambda$ and $N(x, 0; 0) = 0$, $\frac{\partial}{\partial u} N[x, 0; 0] = 0$ for all $x \in \mathbf{R}$.*

Under these hypotheses there exist neighborhoods $\tilde{\Lambda} \subset \Lambda$ of 0 *and $\tilde{U}_1 \subset U \cap \mathcal{X}_1$, $\tilde{U}_2 \subset U \cap \mathcal{X}_2$ of* 0 *and a reduction function $\phi : \mathbf{R} \times \tilde{U}_1 \times \tilde{\Lambda} \to \tilde{U}_2 \subset \mathcal{X}_2$ with the following properties*:

(1) *ϕ is continuously differentiable on $\mathbf{R} \times \tilde{U}_1 \times \tilde{\Lambda}$ up to order k, it and its derivatives are bounded on $\mathbf{R} \times \tilde{U}_1 \times \tilde{\Lambda}$ and for all $x \in \mathbf{R}$*

$$\phi(x, 0; 0) = 0, \qquad \frac{\partial}{\partial u}\phi[x, 0; 0] = 0. \tag{3.8}$$

(2) *The graph*

$$M_C^{\tilde{\lambda}} = \left\{\left(x, u_1 + \phi(x, u_1; \tilde{\lambda})\right) \in \mathbf{R} \times \tilde{U}_1 \times \tilde{U}_2 \;\middle|\; (x, u_1) \in \mathbf{R} \times \tilde{U}_1\right\}$$

is a local integral manifold for (3.5).

(3) *Every solution of* (3.5) *with $\tilde{\lambda} \in \tilde{\Lambda}_0$ and $(u_1(x), u_2(x)) \in \tilde{U}_1 \times \tilde{U}_2$, $x \in \mathbf{R}$, lies completely in $M_C^{\tilde{\lambda}}$.*

Now, we apply this manifold reduction theorem to the system (2.25) with $\mathcal{X} = X$. (3.3) shows that $(A - i\sigma I)^{-1}$ is bounded for large $|\sigma|$. Then A is closed. Since the spectrum of A consists of isolated eigenvalues, we define the spectral projection $\tilde{\mathcal{P}}$ corresponding to the subset $\tilde{\sigma}(A) \cap i\mathbf{R}$ of $\tilde{\sigma}(A)$ by

$$\tilde{\mathcal{P}} w = \frac{1}{2\pi i} \int_C (A - tI)^{-1} w \, dt$$

where C is a closed curve in the resolvent set of A which contains $\tilde{\sigma}(A) \cap i\mathbf{R}$ and no other points of $\tilde{\sigma}(A)$. Write $\mathcal{X}_1 = \tilde{\mathcal{P}}(X)$, $\mathcal{X}_2 = (I - \tilde{\mathcal{P}})(X)$. (3.3) implies (H3) and the smoothness of N gives (H4) by the assumption (A). Note that N does not explicitly include x and is analytic for $\epsilon = 0$. Thus, Theorem 3.1 and the general theory by Mielke [26] imply the following theorem.

Theorem 3.2. (2.25) *has a local integral manifold* $M_C^{\tilde{\lambda}}$ *of class* C^k *in the sense of Theorem* 3.1 *with the assumption* (A)*. Especially,* $M_C^{\tilde{\lambda}}$ *is of class* C^r *for any positive integer* r *as* $\epsilon = 0$*. The reduced system on* $M_C^{\tilde{\lambda}}$ *is periodic in* x *with period* P *and also preserves reversibility, which* (2.25) *has.*

Define an inner product in X as follows

$$(w, w^*) = \int_0^1 \left(h_\tau h_\tau^* + v v^*\right) d\tau + \eta\eta^* + \varrho\varrho^* \tag{3.9}$$

for $w = (h, v, \eta, \varrho)$, $w^* = (h^*, v^*, \eta^*, \varrho^*) \in X$. Then the adjoint operator A^* of A can be obtained using (3.9) and integration by parts.

Lemma 3.2. *The adjoint operator* A^* *of* A *is given by*

$$A^*(h^*, v^*, \eta^*, \varrho^*) = \left(v^* - \frac{\lambda+1}{2b}\varrho^*(\tau-1), -h_{\tau\tau}^*, \frac{\lambda-1}{2b}\varrho^*, \eta^* - h_\tau^*\big|_{\tau=0}\right) \tag{3.10}$$

with a domain $\mathcal{D}(A^*) = \{(h^*, v^*, \eta^*, \varrho^*) \in Y \subset X \mid v^*(0) = -\frac{1}{b}\varrho^*,\ v^*(1) = 0\}$.

4. Normal form

In the following sections, we focus on $(b, \lambda) = (b_0, 1+\mu)$ near E_r for $b_0 > \frac{1}{3}$ and small $\mu > 0$ where $(b_0, 1) \in E_r$ (see Fig. 1). The linear operator A has a double eigenvalue 0. Its eigenvector and generalized eigenvector are given by

$$U_1 = \begin{pmatrix} \tau-1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 0 \\ \tau-1 \\ 0 \\ -1 \end{pmatrix} \tag{4.1}$$

while the eigenvector and generalized eigenvector of eigenvalue 0 corresponding to the adjoint operator A^* are given by

$$U_1^* = \frac{3}{1-3b_0}\begin{pmatrix} -\frac{1}{6}(\tau-1)^3 + \frac{1}{6}(2-3b_0)(\tau-1) \\ 0 \\ \frac{3b_0-1}{6} \\ 0 \end{pmatrix}, \qquad U_2^* = \frac{3}{1-3b_0}\begin{pmatrix} 0 \\ \tau-1 \\ 0 \\ b_0 \end{pmatrix}. \tag{4.2}$$

Moreover,

$$SU_1 = U_1, \qquad SU_2 = -U_2, \qquad SU_1^* = U_1^*, \qquad SU_2^* = -U_2^*,$$
$$\left(U_1, U_1^*\right) = 1, \qquad \left(U_1, U_2^*\right) = 0, \qquad \left(U_2, U_1^*\right) = 0, \qquad \left(U_2, U_2^*\right) = 1.$$

Since the spectrum of A consists entirely of isolated eigenvalues of finite algebraic multiplicity, we can write

$$w = AU_1 + BU_2 + v_2$$

where A and B are real, and v_2 is a linear combination of eigenvectors and generalized eigenvectors corresponding to the rest of eigenvalues. Note that

$$S(A, B) = (A, -B).$$

Now the manifold reduction Theorem 3.2 shows that the set of all bounded solutions of the system (2.25) is determined solely by the ordinary differential equations about A and B

$$\begin{aligned} \dot{A} &= B + f_1(x, A, B; \epsilon, \mu), \\ \dot{B} &= f_2(x, A, B; \epsilon, \mu) \end{aligned} \tag{4.3}$$

where f_i, $i = 1, 2$, are periodic in x with period P and for all $x \in \mathbf{R}$

$$f_i(x, 0, 0; \epsilon, \mu) = 0, \qquad D_A f_i(x, 0, 0; 0, 0) = D_B f_i(x, 0, 0; 0, 0) = 0 \quad \text{for } i = 1, 2,$$
$$S\begin{pmatrix} f_1(x, A, B; \epsilon, \mu) \\ f_2(x, A, B; \epsilon, \mu) \end{pmatrix} = -\begin{pmatrix} f_1(-x, A, -B; \epsilon, \mu) \\ f_2(-x, A, -B; \epsilon, \mu) \end{pmatrix}.$$

We can rewrite f_i for $i = 1, 2$ as the part with ϵ and the part without ϵ

$$f_i(x, A, B; \epsilon, \mu) = g_{i1}(A, B; \mu) + \epsilon g_{i2}(x, A, B; \epsilon, \mu). \tag{4.4}$$

After analyzing the normal form (e.g. see [21]), the system (4.3) reads

$$\begin{aligned} \dot{A} &= B + \mathcal{R}_{11}(A, B; \mu) + \epsilon \mathcal{R}_{12}(x, A, B; \epsilon, \mu), \\ \dot{B} &= P_2(A; \mu) + \mathcal{R}_{21}(A, B; \mu) + \epsilon \mathcal{R}_{22}(x, A, B; \epsilon, \mu) \end{aligned} \tag{4.5}$$

where P_2 is a real polynomial of degree n with a form

$$P_2 = c_1 \mu A + c_2 A^2 + O\big(|A|\big|(\mu, A)\big|^2\big), \quad c_1 = \frac{1}{b_0 - \frac{1}{3}} > 0,\ c_2 = -\frac{1}{2(b_0 - \frac{1}{3})} < 0,$$
$$\mathcal{R}_{i1} = O\big(\big|(A, B)\big|\big|(\mu, A, B)\big|^n\big), \tag{4.6}$$

and $\mathcal{R}_{i2}$ are periodic in x with period P for $i = 1, 2$. Moreover,

$$S\begin{pmatrix} \mathcal{R}_{11}(A, B; \mu) \\ \mathcal{R}_{21}(A, B; \mu) \end{pmatrix} = -\begin{pmatrix} \mathcal{R}_{11}(A, -B; \mu) \\ \mathcal{R}_{21}(A, -B; \mu) \end{pmatrix}, \tag{4.7}$$

$$S\begin{pmatrix} \mathcal{R}_{12}(x, A, B; \epsilon, \mu) \\ \mathcal{R}_{22}(x, A, B; \epsilon, \mu) \end{pmatrix} = -\begin{pmatrix} \mathcal{R}_{12}(-x, A, -B; \epsilon, \mu) \\ \mathcal{R}_{22}(-x, A, -B; \epsilon, \mu) \end{pmatrix}. \tag{4.8}$$

The calculations of c_1 and c_2 are similar to ones in [21]. Theorem 3.2 shows that $\mathcal{R}_{i2}$, $i = 1, 2$, are k-times continuously differentiable with respect to their arguments. By choosing r in Theorem 3.2

sufficiently large when $\epsilon = 0$, one may take n arbitrarily large and let $\mathcal{R}_{i1}$, $i = 1, 2$, have arbitrarily high order derivatives with respect to their arguments.

Let

$$A = -\frac{c_1}{c_2}\mu\tilde{A}, \qquad B = -\frac{c_1^{3/2}}{c_2}\mu^{3/2}\tilde{B}, \qquad x = \frac{1}{\sqrt{c_1\mu}}\tilde{x}$$

which change (4.5) into

$$\begin{aligned} \tilde{A}' &= \tilde{B} + \tilde{\mathcal{R}}_{11}(\tilde{A}, \tilde{B}; \mu) + \epsilon\tilde{\mathcal{R}}_{12}(\tilde{x}, \tilde{A}, \tilde{B}; \epsilon, \mu), \\ \tilde{B}' &= \tilde{A} - \tilde{A}^2 + \tilde{P}_{21}(\tilde{A}; \mu) + \tilde{\mathcal{R}}_{21}(\tilde{A}, \tilde{B}; \mu) + \epsilon\tilde{\mathcal{R}}_{22}(\tilde{x}, \tilde{A}, \tilde{B}; \epsilon, \mu) \end{aligned} \tag{4.9}$$

where the prime $'$ means taking derivatives with respect to $\tilde{x}$,

$$\begin{gathered} \tilde{P}_{21}(\tilde{A}; \mu) = -\frac{c_2}{c_1^2\mu^2}P_2(A; \mu) - \tilde{A} + \tilde{A}^2 = O\big(\mu|\tilde{A}|\big), \\ \tilde{\mathcal{R}}_{11}(\tilde{A}, \tilde{B}; \mu) = -\frac{c_2}{c_1^{3/2}\mu^{3/2}}\mathcal{R}_{11}(A, B; \mu), \qquad \tilde{\mathcal{R}}_{21}(\tilde{A}, \tilde{B}; \mu) = -\frac{c_2}{c_1^2\mu^2}\mathcal{R}_{21}(A, B; \mu), \\ \tilde{\mathcal{R}}_{12}(\tilde{x}, \tilde{A}, \tilde{B}; \epsilon, \mu) = -\frac{c_2}{c_1^{3/2}\mu^{3/2}}\mathcal{R}_{12}(x, A, B; \epsilon, \mu), \\ \tilde{\mathcal{R}}_{22}(\tilde{x}, \tilde{A}, \tilde{B}; \epsilon, \mu) = -\frac{c_2}{c_1^2\mu^2}\mathcal{R}_{22}(x, A, B; \epsilon, \mu) \end{gathered} \tag{4.10}$$

and $\tilde{\mathcal{R}}_{i2}$, $i = 1, 2$, are periodic in $\tilde{x}$ with period $\sqrt{c_1\mu}P$. Symbolically, (4.9) can be written as

$$\tilde{X}' = \tilde{F}(\tilde{X}) + \tilde{P}(\tilde{X}; \mu) + \tilde{\mathcal{R}}_1(\tilde{X}; \mu) + \epsilon\tilde{\mathcal{R}}_2(\tilde{x}, \tilde{X}; \epsilon, \mu) \tag{4.11}$$

where $\tilde{X} = (\tilde{A}, \tilde{B})^T$,

$$\begin{gathered} \tilde{F}(\tilde{X}) = \begin{pmatrix} \tilde{B} \\ \tilde{A} - \tilde{A}^2 \end{pmatrix}, \qquad \tilde{P}(\tilde{X}; \mu) = \begin{pmatrix} 0 \\ \tilde{P}_{21}(\tilde{A}; \mu) \end{pmatrix}, \\ \tilde{\mathcal{R}}_1(\tilde{X}; \mu) = \begin{pmatrix} \tilde{\mathcal{R}}_{11}(\tilde{A}, \tilde{B}; \mu) \\ \tilde{\mathcal{R}}_{21}(\tilde{A}, \tilde{B}; \mu) \end{pmatrix}, \qquad \tilde{\mathcal{R}}_2(\tilde{x}, \tilde{X}; \epsilon, \mu) = \begin{pmatrix} \tilde{\mathcal{R}}_{12}(\tilde{x}, \tilde{A}, \tilde{B}; \epsilon, \mu) \\ \tilde{\mathcal{R}}_{22}(\tilde{x}, \tilde{A}, \tilde{B}; \epsilon, \mu) \end{pmatrix}. \end{gathered} \tag{4.12}$$

Then the problem of the existence of generalized solitary wave solutions of (2.25) is equivalent to that of (4.11). In the following, we concentrate on (4.11).

5. Existence of generalized solitary wave solution

This section proves that (4.11) has a generalized solitary wave solution which approaches a periodic solution as $\tilde{x} \to \pm\infty$. Firstly, we use the method in [23] to obtain that (4.11) has a periodic solution. Then we show that the system $\tilde{X}' = \tilde{F}(\tilde{X})$ has a homoclinic orbit. Using a fixed point theorem and a perturbation method, we prove that this homoclinic orbit persists when higher-order terms are added, which implies the existence of a generalized solitary wave solution.

5.1. Existence of periodic solution

Let

$$\hat{x} = \frac{\tilde{x}}{\sqrt{c_1 \mu}}, \qquad \hat{X}(\hat{x}) = \tilde{X}(\tilde{x})$$

which change (4.11) into

$$\frac{d}{d\hat{x}}\hat{X} = \sqrt{c_1\mu}\big[\tilde{F}(\hat{X}) + \tilde{P}(\hat{X};\mu) + \tilde{\mathcal{R}}_1(\hat{X};\mu) + \epsilon\tilde{\mathcal{R}}_2(\sqrt{c_1\mu}\hat{x}, \hat{X};\epsilon,\mu)\big]. \tag{5.1}$$

Let $H^m_{\mathrm{per}}(0,P)$ be a space of periodic functions of $\hat{x}$ with period P such that their derivatives up to order m are in $L^2(0,P)$, which norm is denoted by $\|\cdot\|_m$. Look for the periodic solution of (5.1) in $H^m_{\mathrm{per}}(0,P) \times H^m_{\mathrm{per}}(0,P)$, which can be expressed as

$$\hat{A}(\hat{x}) = \sum_j A_j \mathrm{e}^{i\frac{2\pi j}{P}\hat{x}}, \qquad \hat{B}(\hat{x}) = \sum_j B_j \mathrm{e}^{i\frac{2\pi j}{P}\hat{x}} \tag{5.2}$$

where $\hat{X} = (\hat{A}, \hat{B})^T$. Plugging (5.2) into (4.11), making the coefficient of each term in the Fourier series equal and using (4.10) and the fixed point theorem, we solve for A_j and B_j, and then for $\hat{A}$ and $\hat{B}$. (More details can be found in [23].)

Theorem 5.1. *Assume that*

$$\epsilon = o\big(\mu^2\big). \tag{5.3}$$

Choose large n such that

$$\mu^{n+1} = O\big(|\epsilon|\mu\big). \tag{5.4}$$

Then the system (5.1) *has a reversible periodic solution* $(\hat{A}^p, \hat{B}^p)^T$ *in* $\hat{x}$ *with period* P *satisfying*

$$\big\|\big(\hat{A}^p, \hat{B}^p\big)\big\|_m \leqslant M\frac{|\epsilon|}{\mu^2} \tag{5.5}$$

where $M > 0$ *is a constant. Thus, the system* (4.11) *has a reversible periodic solution* $(\tilde{A}^p, \tilde{B}^p)^T$ *in* $\tilde{x}$ *with period* $\sqrt{c_1\mu}P$ *satisfying*

$$\big\|\big(\tilde{A}^p, \tilde{B}^p\big)\big\|_m \leqslant M\frac{|\epsilon|}{\mu^{7/4+m/2}}. \tag{5.6}$$

Moreover, the Sobolev embedding theorem yields for $m = 2$

$$\big\|\big(\tilde{A}^p, \tilde{B}^p\big)\big\|_{C^1} \leqslant M\frac{|\epsilon|}{\mu^{11/4}} \tag{5.7}$$

where $\|\cdot\|_{C^1}$ *denotes the norm in* $C^1_B(\mathbf{R})$ *which is a space of continuously differentiable functions with a supremum norm.*

Note that the estimate (5.6) is obtained from (5.5) and the fact $\hat{x} = \frac{\tilde{x}}{\sqrt{c_1\mu}}$. The reversibility of the solution can be obtained by the uniqueness of the solution and the fact that $S(\tilde{A}^p, \tilde{B}^p)(-\tilde{x})$ is also a solution of (4.11). Symbolically, we write this periodic solution $(\tilde{A}^p, \tilde{B}^p)^T(\tilde{x})$ as $\tilde{X}^p_{\epsilon,\mu}(\tilde{x})$.

5.2. Existence of generalized solitary solution

In this subsection, we will prove the following theorem.

Theorem 5.2. *There exists a positive constant μ_0 such that for $0 < \mu \leqslant \mu_0$, if $\epsilon = \epsilon_1\mu^3$ with ϵ_1 bounded,* (4.11) *has a generalized solitary wave solution which is reversible and approaches the periodic solution $\tilde{X}^p_{\epsilon,\mu}(\tilde{x})$ as $\tilde{x} \to \infty$ with the assumption* (A)*.*

Clearly, (5.3) is automatically satisfied under the conditions of this theorem.

The basic idea for the proof of this theorem is similar to the one in [16]. Firstly, we show that the system

$$\tilde{X}' = \tilde{F}(\tilde{X}) \tag{5.8}$$

has a homoclinic solution approaching 0 as $\tilde{x} \to \pm\infty$. Then we prove that (4.11) has a solution exponentially approaching the periodic solution $\tilde{X}^p_{\epsilon,\mu}(\tilde{x})$ as $\tilde{x} \to \infty$. Using the reversibility, this solution can be extended to $\tilde{x} \in (-\infty, 0)$ and approaches $S\tilde{X}^p_{\epsilon,\mu}(-\tilde{x})$ as $\tilde{x} \to -\infty$. The proof is divided into three steps.

Step 1. Homoclinic solution of (5.8).

It is easy to see that (5.8) has a homoclinic solution

$$\tilde{H}(\tilde{x}) = \left(\frac{3}{2}\operatorname{sech}^2\left(\frac{\tilde{x}}{2}\right), -\frac{3}{2}\operatorname{sech}^2\left(\frac{\tilde{x}}{2}\right)\tanh\left(\frac{\tilde{x}}{2}\right)\right)^T \tag{5.9}$$

which exponentially tends to 0 as $\tilde{x} \to \pm\infty$. Moreover, $\tilde{H}$ is reversible, i.e.

$$S\tilde{H}(-\tilde{x}) = \tilde{H}(\tilde{x}). \tag{5.10}$$

Step 2. Solution of (4.11) for $\tilde{x} \in [0, \infty)$.

In order to find a solution of (4.11) near $\tilde{H}(\tilde{x})$, we assume that the solution of (4.11) has a form

$$\tilde{S}(\tilde{x}; \epsilon, \mu) = \tilde{H}(\tilde{x}) + \tilde{Z}(\tilde{x}) + \tilde{X}^p_{\epsilon,\mu}(\tilde{x}) \quad \text{for } \tilde{x} \geqslant 0 \tag{5.11}$$

which exponentially tends to the periodic solution $\tilde{X}^p_{\epsilon,\mu}$ as $\tilde{x} \to \infty$ where $\tilde{Z}$ is a perturbation term to be determined and exponentially tends to 0 as $\tilde{x} \to \infty$. Plugging (5.11) into (4.11) yields the equation about $\tilde{Z}$

$$\tilde{Z}' = \mathcal{L}(\tilde{x})(\tilde{Z}) + N_1(\tilde{x}, \tilde{Z}; \epsilon, \mu) \tag{5.12}$$

where $\mathcal{L}(\tilde{x}) = d\tilde{F}[\tilde{H}(\tilde{x})]$, d means taking the Fréchet derivative, and

$$\begin{aligned} N_1(\tilde{x}, \tilde{Z}; \epsilon, \mu) = {} & \tilde{F}\big(\tilde{H} + \tilde{Z} + \tilde{X}^p_{\epsilon,\mu}\big) - \tilde{F}(\tilde{H}) - d\tilde{F}\big[\tilde{H}(\tilde{x})\big](\tilde{Z}) - \tilde{F}\big(\tilde{X}^p_{\epsilon,\mu}\big) \\ & + \tilde{P}\big(\tilde{H} + \tilde{Z} + \tilde{X}^p_{\epsilon,\mu}; \mu\big) - \tilde{P}\big(\tilde{X}^p_{\epsilon,\mu}; \mu\big) + \tilde{\mathcal{R}}_1\big(\tilde{H} + \tilde{Z} + \tilde{X}^p_{\epsilon,\mu}; \mu\big) - \tilde{\mathcal{R}}_1\big(\tilde{X}^p_{\epsilon,\mu}; \mu\big) \\ & + \epsilon\tilde{\mathcal{R}}_2\big(\tilde{x}, \tilde{H} + \tilde{Z} + \tilde{X}^p_{\epsilon,\mu}; \epsilon, \mu\big) - \epsilon\tilde{\mathcal{R}}_2\big(\tilde{x}, \tilde{X}^p_{\epsilon,\mu}; \epsilon, \mu\big). \end{aligned} \tag{5.13}$$

In the following, we let M be a positive constant. Then N_1 satisfies the following estimates.

Lemma 5.1. *Under the conditions of Theorem* 5.2 *and with bounded $\tilde{Z}$, $\tilde{Z}_1$ and $\tilde{Z}_2$,*

$$\begin{gathered} \big|N_1(\tilde{x}, \tilde{Z}; \epsilon, \mu)\big| \leqslant M\left[\left(\mu + \frac{|\epsilon|}{\mu^{11/4}}\right)\big(e^{-x} + |\tilde{Z}|\big) + |\tilde{Z}|^2\right], \\ \big|N_1(\tilde{x}, \tilde{Z}_1; \epsilon, \mu) - N_1(\tilde{x}, \tilde{Z}_2; \epsilon, \mu)\big| \leqslant M\left(\mu + \frac{|\epsilon|}{\mu^{11/4}} + |\tilde{Z}_1| + |\tilde{Z}_2|\right)\big(|\tilde{Z}_1 - \tilde{Z}_2|\big) \end{gathered} \tag{5.14}$$

for $\tilde{x} \geqslant 0$.

The proof of this lemma is similar to one in [16] by using (4.10) and (5.7).

Obviously, the solution of (5.12) exists if $\tilde{x}$ is in a finite interval and an initial condition is given. In order to prove the existence of solutions for $\tilde{x} \geqslant 0$, we change (5.12) to integral equations and then apply the fixed point theorem to prove the existence of a fixed point of the integral equations.

Now we consider the liner equation of (5.12)

$$\tilde{Z}' = \mathcal{L}(\tilde{x})(\tilde{Z}) \tag{5.15}$$

where

$$\mathcal{L}(\tilde{x}) = \begin{pmatrix} 0 & 1 \\ 1 - 3\,\mathrm{sech}^2(\frac{\tilde{x}}{2}) & 0 \end{pmatrix}. \tag{5.16}$$

The linear operator $\mathcal{L}(\tilde{x})$ has two solutions

$$\begin{aligned} \tilde{s}(\tilde{x}) &= \left(-\frac{3}{2}\,\mathrm{sech}^2\left(\frac{\tilde{x}}{2}\right)\tanh\left(\frac{\tilde{x}}{2}\right), \frac{3}{4}\big(-2 + \cosh(\tilde{x})\big)\,\mathrm{sech}^4\left(\frac{\tilde{x}}{2}\right)\right)^T, \\ \tilde{u}(\tilde{x}) &= \left(\frac{2}{3}\left(\mathrm{sech}^2\left(\frac{\tilde{x}}{2}\right) - \frac{15}{16}\tilde{x}\,\mathrm{sech}^2\left(\frac{\tilde{x}}{2}\right)\tanh\left(\frac{\tilde{x}}{2}\right) - \frac{1}{8}\cosh(\tilde{x})\tanh^2\left(\frac{\tilde{x}}{2}\right) - \tanh^2\left(\frac{\tilde{x}}{2}\right)\right),\right. \\ &\qquad \left. -\frac{1}{192}\,\mathrm{sech}^4\left(\frac{\tilde{x}}{2}\right)\big(120\tilde{x} - 60\tilde{x}\cosh(\tilde{x}) + 185\sinh(\tilde{x}) + 4\sinh(2\tilde{x}) + \sinh(3\tilde{x})\big)\right)^T \end{aligned} \tag{5.17}$$

which have the following properties

$$\tilde{s}(0) = \left(0, -\frac{3}{4}\right)^T, \qquad \tilde{u}(0) = \left(\frac{2}{3}, 0\right)^T, \qquad S\tilde{s}(0) = -\tilde{s}(0), \qquad S\tilde{u}(0) = \tilde{u}(0) \tag{5.18}$$

and

$$\begin{aligned} &\tilde{s}(\tilde{x})e^{\tilde{x}} \to \tilde{s}^\infty \quad \text{as } \tilde{x} \to \infty, \qquad \big|\tilde{s}(\tilde{x})\big| \leqslant Me^{-\tilde{x}} \quad \text{for } \tilde{x} \in [0, \infty), \\ &\tilde{u}(\tilde{x})e^{-\tilde{x}} \to \tilde{u}^\infty \quad \text{as } \tilde{x} \to \infty, \qquad \big|\tilde{u}(\tilde{x})\big| \leqslant Me^{\tilde{x}} \quad \text{for } \tilde{x} \in [0, \infty) \end{aligned} \tag{5.19}$$

where

$$\tilde{s}^\infty = 6(-1,1)^T, \qquad \tilde{u}^\infty = -\frac{1}{24}(1,1)^T.$$

Now, the matrix

$$\mathcal{B}(\tilde{x}) = \big(\tilde{s}(\tilde{x})|\tilde{u}(\tilde{x})\big)$$

is a fundamental matrix for (5.15). Since the trace of $\mathcal{L}(\tilde{x})$ is equal to zero, the $\det(\mathcal{B}(\tilde{x}))$ is independent of $\tilde{x}$, which implies by taking $\tilde{x} \to \infty$,

$$\det\big(\mathcal{B}(\tilde{x})\big) = \det\big(\tilde{s}(\tilde{x})\mathrm{e}^{\tilde{x}}\big|\tilde{u}(\tilde{x})\mathrm{e}^{-\tilde{x}}\big) = \det\big(\tilde{s}^\infty\big|\tilde{u}^\infty\big) = \frac{1}{2}. \tag{5.20}$$

Let the fundamental set of solutions for the adjoint equation of (5.15) be

$$\big\{\tilde{s}^*(\tilde{x}), \tilde{u}^*(\tilde{x})\big\}$$

which is the dual of $\{\tilde{s}(\tilde{x}), \tilde{u}(\tilde{x})\}$ in the sense of the Euclidean inner product on $\mathbf{R}^2$ for each fixed $\tilde{x}$. It follows from (5.19) and (5.20) that for $\tilde{x} \in [0,\infty)$

$$\big|\tilde{u}^*(\tilde{x})\big| \leqslant M\mathrm{e}^{-\tilde{x}}, \qquad \big|\tilde{s}^*(x)\big| \leqslant M\mathrm{e}^{\tilde{x}}. \tag{5.21}$$

The solution of (5.12) that decays to zero at infinity can be found as

$$\tilde{Z} = \int_0^{\tilde{x}} \big\langle N_1(t,\tilde{Z};\epsilon,\mu), \tilde{s}^*(t)\big\rangle\,dt\,\tilde{s}(\tilde{x}) - \int_{\tilde{x}}^{\infty} \big\langle N_1(t,\tilde{Z};\epsilon,\mu), \tilde{u}^*(t)\big\rangle\,dt\,\tilde{u}(\tilde{x})$$

or

$$\tilde{Z} = \mathcal{G}(\tilde{Z};\epsilon,\mu) \tag{5.22}$$

where $\langle\cdot\rangle$ denotes the Euclidean inner product on $\mathbf{R}^2$.

Choose $\nu \in (\frac{1}{2}, 1)$ and consider (5.22) as a fixed point problem in a Banach space

$$E_\nu = \Big\{\tilde{Z} \in C(0,\infty)\ \Big|\ \sup_{\tilde{x}\in[0,\infty)} \big\{\big|\tilde{Z}(\tilde{x})\big|\mathrm{e}^{\nu\tilde{x}}\big\} < \infty\Big\}$$

with the norm

$$\|\tilde{Z}\|_\nu = \sup\big\{\big|\tilde{Z}(\tilde{x})\big|\mathrm{e}^{\nu\tilde{x}}\ \big|\ \tilde{x} \in [0,\infty)\big\},$$

which implies that $\tilde{Z}$ exponentially tends to zero as $\tilde{x} \to \infty$. It is easy to obtain the following lemma using Lemma 5.1, (5.19) and (5.21).

Lemma 5.2. *Under the conditions of Theorem* 5.2, *the function* $\mathcal{G}$ *satisfies*

$$\begin{aligned} \left\|\mathcal{G}(\tilde{Z};\epsilon,\mu)\right\|_\nu &\leqslant M\left[\|\tilde{Z}\|_\nu^2+\left(\mu+\frac{|\epsilon|}{\mu^{11/4}}\right)\left(1+\|\tilde{Z}\|_\nu\right)\right], \\ \left\|\mathcal{G}(\tilde{Z}_1;\epsilon,\mu)-\mathcal{G}(\tilde{Z}_2;\epsilon,\mu)\right\|_\nu &\leqslant M\left[\mu+\frac{|\epsilon|}{\mu^{11/4}}+\|\tilde{Z}_1\|_\nu+\|\tilde{Z}_2\|_\nu\right]\|\tilde{Z}_1-\tilde{Z}_2\|_\nu \end{aligned} \tag{5.23}$$

for $\tilde{x}\geqslant 0$ *and* $\tilde{Z},\tilde{Z}_1,\tilde{Z}_2\in E_\nu$.

Now we apply the contraction mapping theorem to find a fixed point of $\mathcal{G}$. Let $\bar{B}_{\tilde{r}}(0)\subset E_\nu$ be a closed ball in E_ν with a radius $\tilde{r}$. If we choose $\mu>0$ small enough such that

$$M\left[\tilde{r}^2+\left(\mu+\frac{|\epsilon|}{\mu^{11/4}}\right)(1+\tilde{r})\right]\leqslant\tilde{r},\qquad M\left[\mu+\frac{|\epsilon|}{\mu^{11/4}}+2\tilde{r}\right]\leqslant\frac{1}{2}, \tag{5.24}$$

then $\mathcal{G}$ is a contraction in $\bar{B}_{\tilde{r}}(0)$. The conditions in (5.24) are easily satisfied if $\tilde{r}=\mu^{1/8}$ under the assumption $\epsilon=\epsilon_1\mu^3$ in Theorem 5.2. Thus, by the contraction mapping theorem, (5.22) has a unique solution $\tilde{Z}(\tilde{x};\epsilon,\mu)$ satisfying

$$\left\|\tilde{Z}(\tilde{x};\epsilon,\mu)\right\|_\nu\leqslant\tilde{r}=\mu^{1/8}. \tag{5.25}$$

By differentiating (5.22) with respect to $\tilde{x}$ and using the same argument as that for (5.25) and an extension of a contraction mapping principle in [29], we can show that $\tilde{Z}_{\tilde{x}}(\tilde{x};\epsilon,\mu)$ exists and satisfies

$$\left\|\tilde{Z}_x(\tilde{x};\epsilon,\mu)\right\|_\nu\leqslant\tilde{r}=\mu^{1/8}. \tag{5.26}$$

Similarly, we can prove that $\hat{Z}$ is smooth in its arguments. If we want to get the higher smoothness of $\tilde{Z}$, we have to take large k in the assumption (A) and ϵ enough small by (5.6) and (5.7).

Step 3. Solution of (4.11) for $\tilde{x}\in(-\infty,0]$.

(5.10), (5.18) and (5.22) yield that

$$S\big(\tilde{S}(0;\epsilon,\mu)\big)=\tilde{S}(0;\epsilon,\mu) \tag{5.27}$$

since $\tilde{X}_{\epsilon,\mu}(\tilde{x})$ is reversible. To construct the solution for $\tilde{x}<0$, we know from the reversibility of the system (4.11) that both $\tilde{S}(\tilde{x};\epsilon,\mu)$ and $S(\tilde{S}(-\tilde{x};\epsilon,\mu))$ are solutions of (4.11) and at $\tilde{x}=0$

$$S\big(\tilde{S}(0;\epsilon,\mu)\big)=\tilde{S}(0;\epsilon,\mu).$$

Thus, by the uniqueness of the solution for an initial value problem, we can define a solution of (4.11) as

$$\tilde{S}_1(\tilde{x})=\begin{cases}\tilde{S}(\tilde{x};\epsilon,\mu) & \text{for } \tilde{x}\geqslant 0,\\ S(\tilde{S}(-\tilde{x};\epsilon,\mu)) & \text{for } \tilde{x}\leqslant 0.\end{cases}$$

Then $S\tilde{S}_1(-\tilde{x})=\tilde{S}_1(\tilde{x})$. Thus, the solution $\tilde{S}_1(\tilde{x})$ of (4.11) is a reversible homoclinic connection to the periodic solution $\tilde{X}_{\epsilon,\mu}(\tilde{x})$ as $\tilde{x}\to\infty$ and the periodic solution $S\tilde{X}_{\epsilon,\mu}(-\tilde{x})$ as $\tilde{x}\to-\infty$. This completes the proof of Theorem 5.2.

Remark 5.1.

(1) By the relation $\mu^{n+1} = O(|\epsilon|\mu)$ between ϵ and μ in (5.4), we may choose a suitable n such that (5.4) is satisfied under the conditions in Theorem 5.2. For example, take $\epsilon_1 = 1$ and $n = 3$ with $\epsilon = \mu^3$.
(2) Using the above method, we can also prove the existence of a generalized solitary wave of (2.25) approaching a periodic solution when (b, λ) is near other curves in Fig. 1. This method can also be used for waves past other disturbances like a small bump with compact support.

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